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1. INTRODUCTION: THE PROBLEM OF APPORTIONMENT. Ten goats are to be assigned to three brothers in numbers proportional to the ages (in years) of the recipients. Given the integral nature of a goat, it is not generally possible to meet exactly the condition of proportionality, and the resulting "apportionment problem" is a classic of operational research. The associated literature is extensive, including on the one hand discussions of criteria to be used in assessing different schemes, and on the other hand accounts of the properties of specific classes of scheme. A point of especial focus has been the apportionment of the seats in the House of Representatives to the states of the U.S.A. There are currently fifty states (excluding the District of Columbia) and 435 seats, which are to be divided among the states according to the U.S. Constitution [Article I, Section 2] of 1787 thus:

> Representatives ... shall be apportioned among the several States ... according to their respective Numbers ....

No scheme is proposed in the Constitution, and Art. I therein permits a spectrum of interpretation of the phrase "according to their respective numbers." Politicians, lawyers, mathematicians, and others have been involved ever since in the cyclical debate of how to apportion the seats.

Although there is little in this article that is specific to the U.S. Congress, for ease of exposition we shall use the terminology of the last problem. Our targets here are to survey the general area and to propose a new method of apportionment that, in a certain way to be made more precise, meets all the usual criteria for such schemes. This new method is a lottery scheme whose implementation uses (pseudo-)random numbers. The scheme is fair in the sense of expectations so long as no minimal number of seats need be allocated to each state however small. In the Congressional example, the Constitution contains an additional condition that every state shall receive at least one seat. It is this violation of the principle of proportional representation that renders futile all attempts to obtain a truly fair system in which individuals are equally represented.

In section 3 we describe a new method that meets the so-called quota condition (see section 2) and gives proportional representation. In section 4 we present an adaptation of this method for use in situations in which there are lower bounds on the allocations sought. We shall see in section 4 that, in the presence of a lower bound, the quota condition (see section 2) provides another source of unfairness.

The established theory of deterministic apportionment is summarized in section 5. Those interested in learning more of the history and practice of apportionment should consult the book of Balinski and Young [**1**].

2. QUOTA. We suppose that there are s states with respective populations $\pi_1, \pi_2, \dots, \pi_s$ and that there are r seats in the House of Representatives. The total population size is $\Pi = \pi_1 + \pi_2 + \cdots + \pi_s$, and thus the exact *quota* of seats for state i is the number $q_i = r\pi_i/\Pi$. The problem is that the q_i are not generally integers, whereas representatives are (by axiom) indivisible. We call $\pi = (\pi_1, \pi_2, \dots, \pi_s)$ and $q = (q_1, q_2, \dots, q_s)$ the *population vector* and *quota vector*, respectively. We refer to the pair (π, r) as a *problem*.

An *allocation* is a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ of nonnegative integers with sum r. An allocation α is said to *satisfy quota* if $\alpha_i \in \{\lfloor q_i \rfloor, \lceil q_i \rceil\}$ for all i. An allocation is said to *violate quota* if it does not satisfy quota. Here $\lfloor x \rfloor$ denotes the greatest integer not exceeding x, and $\lceil x \rceil$ denotes the least integer not less than x. We speak of $\lfloor q_i \rfloor$ (respectively, $\lceil q_i \rceil$) as the *lower* (respectively, *upper*) *quota* for state i.

It seems generally (if not universally) accepted that the property of satisfying quota is desirable. It is clear that, for all problems of the foregoing type, there exists necessarily at least one allocation that satisfies quota. This can cease to be the case when further conditions are added. In a variety of situations, including that of the U.S. Congress, there is a requirement that the α_i not be too small. Let $l = (l_1, l_2, \ldots, l_s)$ be a given vector of nonnegative integers. We say that an allocation α *has lower bound* l if $\alpha_i \geq l_i$ for all i. Of special interest is the case $l = 1$, the vector of ones, which is the lower bound specified in Art. I of the U.S. Constitution:

> each State shall have at Least one Representative ....

Such a requirement is potentially disturbing since there exist problems (π, r) for which no allocation exists satisfying quota and having lower bound 1. For a simple (if extreme) example, consider the case when $\pi = (1, 1, 7)$ and $r = 3$.

It is a simple matter to see that there exists an allocation that satisfies quota and has lower bound l if and only if

$$l_i \leq \lceil q_i \rceil$$

for all i, and

$$\sum_i \max\{l_i, \lfloor q_i \rfloor\} \leq r.$$

The satisfying of quota seems to be regarded as paramount among properties of allocations. As noted earlier, we shall see in section 4 that, in the presence of a lower bound, the requirement of quota can be a further source of unfairness. There are certain desirable properties of schemes for finding allocations, most prominently that the scheme avoids each of the three "paradoxes"—Alabama, population, and new-state—to which we return in section 5.

A word for the novice—in common with other similar problems, one may be tempted to identify many desirable properties of schemes, only to find that no scheme has them all. A well-known example of this phenomenon is the result known as Arrow's Impossibility Theorem, which states that no preference ranking exists for a society that embraces a certain collection of five reasonable axioms (see [**4**, chap. 14]).

3. STOCHASTIC APPORTIONMENT WITHOUT LOWER BOUNDS. Although lottery schemes have been mentioned briefly in the literature (see, for example, [**1**]), the established theory is concentrated on deterministic schemes. Our purpose here is to propose a family of stochastic schemes, one in particular, that satisfy quota and that have the advantage of being truly fair and proportional in that each state receives a (possibly random) number of seats having mean value equal to the quota of the state. We shall see in section 4 that this cannot generally be achieved in the presence of lower bounds on allocations, and therefore *we make the facilitating assumption for the duration of this section that no lower bound is required.*

A *random allocation* is a vector $A = (A_1, A_2, \ldots, A_s)$ of nonnegative-integer-valued random variables with sum r. A *randomized scheme* is a mapping that, to each problem (π, r), allocates a probability distribution on the space of appropriate allocations. Otherwise expressed, a randomized scheme results in a random allocation (we shall not spend any time on the choice of probability space, and such like).

There is a subtlety to the notion of a randomized scheme that we discuss briefly. We may seek to apply such a scheme to two given problems, perhaps by applying it twice to the same problem (π, r), or perhaps by applying it to (π, r) and to another problem (π', r') obtained from (π, r) by changing some of the parameters. In so doing, we encounter the question of "coupling." That is, since randomized schemes make use of pseudo-random numbers, we shall need to specify whether, at the one extreme, we re-use for the second problem the pseudo-random numbers used already for the first, or, at the other extreme, we make use of "new" pseudo-random numbers. We discuss this no further at this stage (see, however, the two final paragraphs of this section), since the discussion will concentrate for the moment on the use of randomized schemes for a single problem only.

We say that a random allocation A *satisfies quota almost surely* if

$$\mathbb{P}(A \text{ satisfies quota}) = 1.$$

Here $\mathbb{P}$ denotes probability, and (later) $\mathbb{E}$ denotes expectation. A randomized scheme is said to "satisfy quota" if the ensuing random allocation satisfies quota almost surely.

In advance of making a concrete proposal for a randomized scheme, we point out that there exist in general a multiplicity of such schemes that satisfy quota. Let $q'_i = q_i - \lfloor q_i \rfloor$, let $s' = |\{i : q'_i > 0\}|$, and let $r' = r - \sum_i \lfloor q_i \rfloor$. We call $q' = (q'_i : 1 \le i \le s)$ the *fractional-quota vector*. Since state i receives $\lfloor q_i \rfloor$ seats of right, we are required only to allocate the remaining r' seats among the s' "unsatisfied" states. This may be done in $\binom{s'}{r'}$ ways, to each of which we must assign a probability. If we require in addition that the scheme be *fair* in the sense that the ensuing allocation A satisfies $\mathbb{E}(A_i) = q_i$ for all i, then we obtain thus a set of s' constraints. Thus the space of such randomized schemes may have up to $\binom{s'}{r'} - s'$ degrees of freedom. In the special cases $r' = 1$ and $r' = s' - 1$, there is a unique randomized scheme that is fair and satisfies quota.

We turn now to our concrete proposal. There are three steps.

I. We permute at random the labels of the states.

   This is proposed since the scheme that follows depends on the labelling of the states (that is, on the indices of the π_i), and it seems desirable to reduce to a minimum any correlations that depend on this extraneous element. In the Congressional example, it would arguably not be right for the allocation to Alabama to depend systematically to a greater degree on that to Alaska than that to Maine. *For ease of presentation, in the following we do not change the notation π_i, but assume that π is the population vector after permuting at random.*

II. We provisionally allocate $\lfloor q_i \rfloor$ seats to state i.

   This is prompted by the minimal quota for each state, and leaves so far unallocated a certain number $r' = r - \sum_i \lfloor q_i \rfloor$ of seats. Let $q'_i = q_i - \lfloor q_i \rfloor$, as before.

III. Let U be a random variable with the uniform distribution on [0, 1], and let $Q_i = U + \sum_{j=1}^{i} q'_j$. Let A'_i be the indicator function of the event that the interval $[Q_{i-1}, Q_i)$ contains an integer. We allocate a further A'_i seats to state i.

State i receives a total of $\lfloor q_i \rfloor + A'_i = A_i$ seats. The total number of seats allocated at step III equals the length of the interval $[Q_0, Q_s)$, which is $\sum_i \{q_i - \lfloor q_i \rfloor\} = r'$, whence A is an allocation (that is, it has sum r). It is evident that $\lfloor q_i \rfloor \le A_i \le \lceil q_i \rceil$ with probability one, whence we deduce that A satisfies quota. In addition, each Q_j, when reduced modulo 1, is uniformly distributed on the interval [0, 1]. Therefore, $\mathbb{E}(A'_i)$ is just the length of the interval $[Q_{i-1}, Q_i)$, which is to say that $\mathbb{E}(A'_i) = Q_i - Q_{i-1} = q'_i$, and hence that $\mathbb{E}(A_i) = q_i$. We summarize this by saying that the scheme satisfies quota and is fair in the sense that the mean number of seats per head of population is constant between states.

Henceforth we use the word "fair" only in this sense: a stochastic allocation scheme is termed *fair* within a given subset T of states if the mean number of seats per head of population is a constant for all states in T. We suppress reference to the set T when it is the set of all states.

We offer no justification for this scheme apart from fairness and ease of implementation. We caution against adopting any randomized scheme without a *proof* of fairness, and as an example we summarize one arguably reasonable but definitely unfair scheme for allocating the remaining r' seats.

Conditional sampling. We select independent indices I_t, $1 \le t \le r'$, each I_t having distribution $\mathbb{P}(I_t = i) = q'_i / \sum_i q'_i$, and we consider a random vector

$$J = (J_1, J_2, \dots, J_{r'})$$

having the probability distribution of $I_1, I_2, \dots, I_{r'}$ conditional on $I_u \ne I_v$ for $u \ne v$. (See [**3**] for an account of conditional probability.) We allocate one extra seat to those states having indices $J_1, J_2, \dots, J_{r'}$. It is the case that

$$\mathbb{E}(|\{t : I_t = i\}|) = q'_i$$

for all i, but the same statement is generally false with I replaced by J. This is easily seen when r' and s' are large. The conditioning amounts to a large deviation, and the distribution of J may be close to an appropriately tilted distribution (see [**3**, sec. 5.11]).

We close this section with some remarks on the use of pseudo-random numbers. Some will argue that the degree of fairness of a stochastic scheme may not be evident to the population, and politicians also may be unwilling to accept such a scheme, since politicians are very sensitive to the marginal value to a party of a single seat. However, lotteries are already in wide use in areas having impact on individuals, not least in state-accredited systems for raising money for so-called good causes. Moreover, the allocation of individuals to the control group of a medical trial is usually done by lottery, and such a decision may be a matter of life or death. See [**6**] for an extended discussion of the drawing of lots.

When applying a randomized scheme to two or more problems, one needs to decide whether or not to resample the required pseudo-random numbers. The expectations under study remain the same, but the external perception of fairness is likely to be greater if the roulette wheel is spun afresh. A statistical virtue of resampling is the reduction of variances.

4. STOCHASTIC APPORTIONMENT WITH LOWER BOUNDS. We consider in this section the problem of finding an allocation that satisfies quota and is subject to a lower bound l. Let $l = (l_1, l_2, \dots, l_s)$ be a vector of nonnegative integers satisfying $l_i \le \lceil q_i \rceil$ for all i. We assume in addition that $\sum_i l_i \le r$, since otherwise there exists no allocation with lower bound l and sum r. We call a state *small* if $q_i < l_i$, and we note that the lower-bound condition is tantamount to favouring the small states. It follows that, whenever there exist small states, the other states are at a disadvantage, and that there can exist no scheme that is fair to all states. However, can we find a randomized scheme that is fair within large groups of states?

Let $I_- = \{i : q_i < l_i\}$ be the set of small states, and also $I_= = \{i : q_i = l_i\}$ and $I_+ = \{i : q_i > l_i\}$. We assume that there exists at least one small state, in that $I_- \neq \varnothing$, since otherwise there is no new problem. Since $\sum_i l_i \leq r$, it must be the case that $I_+ \neq \varnothing$. For $i \in I_- \cup I_=$, we allocate to state i exactly l_i seats, and we write $\mu = r - \sum_{i \in I_- \cup I_=} l_i$ for the number of remaining seats. Note that $\sum_{i \in I_+} q_i > \mu$, whence no fair allocation can exist.

In allocating seats to states in I_+, we seek a guiding principle, and we propose the principle of *equality of representation among states in* I_+. That is, we seek a scheme that is fair within I_+.

The scheme of section 3 (particularly step III thereof) is based on the quota vector q and the fractional-quota vector q'. Since the allocations to states in $I_- \cup I_=$ have already been determined, we are directed by the proposed principle to seek a new quota vector $Q = (Q_i : i \in I_+)$ such that:

(a) $\sum_{i \in I_+} Q_i = \mu$;

(b) the mean number of representatives per head of population is constant among states in I_+, which is to say that there exists a constant γ such that

$$\frac{Q_i}{q_i} = \gamma \tag{1}$$

for all $i \in I_+$;

(c) $\lfloor q_i \rfloor \leq Q_i \leq \lceil q_i \rceil$ for all $i \in I_+$.

Following consideration of the lower bound condition, we shall require also that

(d) $Q_i \geq l_i$ for all $i \in I_+$.

Note that (c) implies (d) since, by (c), $Q_i \geq \lfloor q_i \rfloor \geq l_i$ for $i \in I_+$.

We next investigate conditions under which (a), (b), and (c) may be achieved simultaneously. Assume that (a) and (b) hold. We sum (1) over $i \in I_+$ to obtain by (a) that

$$\gamma = \frac{\sum_{i \in I_+} Q_i}{\sum_{i \in I_+} q_i} = \frac{\mu}{\sum_{i \in I_+} q_i}.$$

It follows that (c) is satisfied if and only if $Q_i = \gamma q_i \geq \lfloor q_i \rfloor$ for all $i \in I_+$.

If (c) holds, which in the circumstances is equivalent to $\lfloor Q_i \rfloor = \lfloor q_i \rfloor$ for $i \in I_+$, then we proceed by applying the algorithm of the last section thus. For $i \in I_+$, we allocate $\lfloor q_i \rfloor$ seats to state i. Then we apply step III of the algorithm to the new fractional-quota vector $Q' = (Q_i - \lfloor Q_i \rfloor : i \in I_+)$ with (by (a)) $\mu - \sum_{i \in I_+} \lfloor q_i \rfloor$ seats. The outcome is a random allocation that satisfies quota and that is fair within I_+.

Suppose that (c) does not hold, in that there exists $i \in I_+$ such that $Q_i < \lfloor q_i \rfloor$. One strategy would be to apply the scheme of the last section to the quota vector Q, obtaining thereby a random allocation A satisfying $\mathbb{E}(A_i) = Q_i$ for each $i \in I_+$. If this random allocation does not satisfy quota, then the scheme has failed. There is however a certain probability that quota is satisfied. By counting the mean number of states whose allocations do not satisfy the respective quotas, we find that

$$\mathbb{P}(A \text{ does not satisfy quota}) \leq \sum_{i \in I_+} \max\{1, \lfloor q_i \rfloor - Q_i\} 1_{\{Q_i < \lfloor q_i \rfloor\}}, \tag{2}$$

where 1_A denotes the indicator function of the event A. Equality holds in (2) if there exists at most one state i with $Q_i < \lfloor q_i \rfloor$.

Let us suppose that (c) does not hold but that the outcome A satisfies quota. A slightly subtle point is that, conditional on this event, A is not fair. This is so because the conditioning changes the expectations. In summary, the scheme—*work with the quota vector Q repeatedly until one obtains a random allocation that satisfies quota*—is not a fair scheme.

A feasible line of enquiry that we have not pursued here is to postulate probabilistic models for problems, and to calculate the probability for such a model that the scheme we have devised results in an allocation that does not satisfy quota. In certain circumstances, the "law of anomalous numbers" (see [**2**], [**3**]) could be used as part of a basis for such a model.

If the satisfying of quota is paramount, then one must accept a further degree of unfairness, and we propose the following. If $Q_i < \lfloor q_i \rfloor$, we allocate to state i exactly $\lfloor q_i \rfloor$ seats. Writing $J_+ = \{i \in I_+ : Q_i \geq \lfloor q_i \rfloor\}$, we now iterate the foregoing process restricted to the states in J_+. That is, we seek, as earlier, numbers R_i for $i \in J_+$ such that:

(a′) $\sum_{i \in J_+} R_i = \mu - \sum_{i \in I_+ \setminus J_+} \lfloor q_i \rfloor$;

(b′) the mean number of representatives per head of population is constant among states in J_+, in that there exists a constant γ' such that $R_i/q_i = \gamma'$ for all $i \in J_+$;

(c′) $\lfloor q_i \rfloor \leq R_i \leq \lceil q_i \rceil$ for all $i \in J_+$.

We find this time from (a′) and (b′) that $R_i = \gamma' q_i$, where

$$\gamma' = \frac{\mu - \sum_{i \in I_+ \setminus J_+} \lfloor q_i \rfloor}{\sum_{i \in J_+} q_i}.$$

If $R_i \geq \lfloor q_i \rfloor$ for all $i \in J_+$, we apply the scheme to the new quota vector $R = (R_i : i \in J_+)$. Otherwise, we note that any state i with $R_i < \lfloor q_i \rfloor$ must be handled in a way that will disfavour those remaining, namely, by allocating to it $\lfloor q_i \rfloor$ seats.

This scheme, when iterated to reach a conclusion, identifies different levels of unfairness in its consecutive applications of the principle of *equality of representation among the remaining states*. Note that it terminates with a random allocation that satisfies quota if and only if there exists an allocation that both satisfies quota and obeys the lower bound.

We emphasize that, in the presence of a lower bound, the principle of satisfying quota is another potential source of unfairness. The upper quota presents no problem, but the lower quota can indeed be problematic.

Let us apply the above argument in the Congressional example, with $l = 1$. In the following table are listed those states i that in the nine ten-yearly apportionments of 1920–2000 have $Q_i < \lfloor q_i \rfloor$, these being the apportionments with the current House size of 435. In just four of these apportionments (namely, those of 1920, 1950, 1970, and 2000) do there exist states i with $Q_i < \lfloor q_i \rfloor$, and in each such case there is only a small probability that the proposed scheme results in an allocation based on Q that does not satisfy quota. For example, in 1950 the ensuing allocation could only fail to satisfy quota if New York were allocated forty-two seats rather than forty-three, an event of probability $\lfloor q_i \rfloor - Q_i = 43 - 42.962 = 0.038$, where i is the index of New York. The corresponding probability for the year 2000 is $\lfloor q_j \rfloor - Q_j = 19 - 18.999 = 0.001$, where j is the index of Pennsylvania. A similar calculation is valid for the other years, and in each case the resulting probability is small. Based on this empirical evidence, we claim that the proposed scheme, based on Q, is likely to result in an apportionment of the House of Representatives that satisfies quota.

Table 1. For the apportionments of 1920–2000, this is a list of states i for which $Q_i < \lfloor q_i \rfloor$. The entries are derived from data to be found in [**1**].

Year	# small states	States with $Q_i < \lfloor q_i \rfloor$	Original quota q_i	New quota Q_i
1920	3	Pennsylvania	36.053	35.973
		Florida	4.004	3.995
1930	3	none		
1940	3	none		
1950	3	New York	43.038	42.962
1960	4	none		
1970	3	Virginia	10.000	9.984
1980	3	none		
1990	3	none		
2000	4	Pennsylvania	19.013	18.999

However, in the years in question, such an outcome would be slightly unfair to the other states. Suppose that, instead, we acknowledge this unfairness explicitly by automatically allocating to any state i with $Q_i < \lfloor q_i \rfloor$ the number $\lfloor q_i \rfloor$ seats. In each case, when we calculate the amended quota vector $R = (R_i : i \in J_+)$, we find that every R_i satisfies $R_i \geq \lfloor q_i \rfloor$. The outcome of the composite scheme is therefore an allocation that favours firstly the small states and secondly any state i with $Q_i < \lfloor q_i \rfloor$, and then proceeds to allocate seats to the remaining states in a way that gives equality of representation between them.

Finally we note the existence of an alternative method that might in principle be adopted. This is easier to apply but violates the criterion of fairness. It is as follows. One reduces the original fractional-quota vector $(q_i' : i \in I_+)$ by a constant factor, that is, one works with a new fractional-quota vector $(q_i'' : i \in I_+)$ given by $q_i'' = \delta q_i'$, where δ is chosen in such a manner that

$$\sum_{i \in I_+} \{\lfloor q_i \rfloor + q_i''\} = \mu.$$

5. DETERMINISTIC APPORTIONMENT. It is conventional to apportion the House of Representatives using an algorithm that is deterministic rather than stochastic. The book [**1**] is an account of theory and practice for mathematicians and non-mathematicians alike, and the reader is referred to it for a full account. No attempt is made here to do more than summarize the situation.

Let us assume that there is no lower bound. The discussion in the literature has concentrated largely on a category of schemes termed *divisor methods*. Let $\delta : \{0, 1, 2, \ldots\} \to \mathbb{R}$ be a given function satisfying $b \leq \delta(b) \leq b + 1$ for all b. Let λ be a parameter taking positive values, to be thought of as the notional number of head of population to be represented by each representative. The population π_i of state i is divided by λ to obtain the λ-*quota* π_i/λ of seats for that state. At the first stage, we find the λ-*allocation* to state i, defined as

$$\alpha_i(\lambda) = \begin{cases} \lfloor \pi_i/\lambda \rfloor & \text{if } \pi_i/\lambda \le \delta(\lfloor \pi_i/\lambda \rfloor), \\ \lfloor \pi_i/\lambda \rfloor + 1 & \text{if } \pi_i/\lambda > \delta(\lfloor \pi_i/\lambda \rfloor). \end{cases}$$

When equality holds in that $\pi_i/\lambda = \delta(\lfloor \pi_i/\lambda \rfloor)$, we have set $\alpha_i(\lambda) = \lfloor \pi_i/\lambda \rfloor$ for the sake of being definite, but it is important only that some clear rule be followed. The resulting λ-allocation $\alpha(\lambda) = (\alpha_i(\lambda) : 1 \le i \le s)$ does not generally sum to the house size r. At the second stage, we "tune" λ until we find $\overline{\lambda}$ such that $\sum_i \alpha_i(\overline{\lambda}) = r$, and we output the allocation $\alpha(\overline{\lambda})$.

There is an infinity of possible choices for the function δ, of which the instances in Table 2 have been studied.

Table 2. The five principal divisor methods ordered by increasing δ.

Name	Originator	The function $\delta(b)$
Smallest divisors	Adams (1832)	b
Harmonic means	Dean (1832)	$2/\{b^{-1} + (b+1)^{-1}\}$
Equal proportions	Hill (1911)	$\sqrt{b(b+1)}$
Major fractions	Webster (1832)	$b + \frac{1}{2}$
Greatest divisors	Jefferson (1792)	$b + 1$

A sixth scheme, termed the method of Hamilton (1792) or the "method of largest remainders", is as follows. A state with quota q_i receives of right $\lfloor q_i \rfloor$ seats. The remaining $r - \sum_i \lfloor q_i \rfloor$ seats are allocated to those $r - \sum_i \lfloor q_i \rfloor$ states j with largest remainders $q_j - \lfloor q_j \rfloor$.

One seeks principles that enable distinctions to be drawn between these six schemes. In common with other instances in operational research, one can be over-principled. Every method has its drawbacks, and to seek the perfect system can be to eliminate all possibilities. A detailed and informative discussion is to be found in [**1**], from which a few points are extracted here.

A scheme is called *population monotone* if: when individuals move from state i to state j (where $i \neq j$), i does not receive thereby more seats *and* j fewer. We learn from [**1**, Theorem 6.1] that no scheme exists that invariably is population monotone and satisfies quota. Furthermore (see [**1**, Proposition 6.4]), Jefferson's "greatest divisor" method is the only population monotone method that invariably stays above lower quota (in that the ensuing allocation α satisfies $\alpha_i \ge \lfloor q_i \rfloor$ for all i), and Adams's "smallest divisor" method is the only population monotone scheme that invariably stays below upper quota (in that $\alpha_i \le \lceil q_i \rceil$ for all i).

A point of focus is the degree to which a scheme favours large over small states. There are various ways of measuring such bias, both theoretical and empirical, and the case is made in [**1**] and [**7**] that Webster's "major fraction" method is the least biased in this regard.

A scheme is said to be *house monotone* if no state's allocation diminishes when the size r of the house increases. A scheme that is not house monotone is said to suffer from the *Alabama paradox*. It is considered desirable that a scheme be house monotone. Other desirable features of schemes include the absence of what are known as the *population paradox* (in a growing population, state i can grow faster than state j, and yet lose a seat to j), and the *new-states paradox* (a new state may join the union, with an appropriate number of new seats, but the allocations to the original states change).

One may ask in what sense the stochastic apportionment scheme of section 3 meets these requirements. The latter scheme is population monotone in a stochastic sense: when state i loses people to state j, the number of seats allocated to i (respectively, j) is stochastically nonincreasing (respectively, nondecreasing). (See [**3**, sec. 4.12] for a definition of stochastic ordering.) In a similar stochastic sense, the scheme is house monotone.

Finally, we consider the case of deterministic schemes in situations where there is a nontrivial lower bound on the allocation sought. As stated already, there may exist no allocation that satisfies both quota *and* the lower bound, and even if there exists such an allocation, there will generally exist no scheme that is fair across the board. The method used currently for apportioning the House of Representatives is to allocate one seat to each state however small (in 2000 there were just four states whose quotas were smaller than one) and then to apply the method of equal proportions to those states whose "residual" quotas are strictly positive. The outcomes of this scheme have, fortunately in recent decades, been allocations that have satisfied quota.

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REFERENCES

1. M. L. Balinski and H. P. Young, *Fair Representation, Meeting the Ideal of One Person, One Vote*, 2nd ed., The Brookings Institution, Washington, D.C., 2001.
2. W. Feller, *An Introduction to Probability Theory and its Applications*, vol. 2, 2nd ed., John Wiley & Sons, New York, 1971.
3. G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes*, 3rd ed., Oxford University Press, Oxford, 2001.
4. R. D. Luce and H. Raiffa, *Games and Decisions*, Dover, New York, 1989.
5. F. Pukelsheim, Pukelsheim's proportional representation literature list, www.uni-augsburg.de/bazi/pprll.html.
6. W. A. Silverman and I. Chalmers, Casting and drawing lots, in *Controlled Trials from History*, I. Chalmers, I. Milne, and U. Tröhler, eds., 2001, www.jameslindlibrary/essays/casting_of_lots/casting.html.
7. H. P. Young, *Dividing the House: Why Congress Should Reinstate an Old Apportionment Formula*, Policy Brief no. 88, The Brookings Institution, Washington, D.C., 2001; available at www.brook.edu/dybdocroot/comm/policybriefs/pb88.htm.

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